On some inverse conductivity recovery problem in a sphere:
Uniqueness and reconstruction results with applications to EEG

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Outline

1. Motivation
2. Spherical Model
3. Uniqueness results
4. Reconstruction algorithm
5. Numerical and stability analysis
Conductivity estimation

- Reconstruct the conductivity of the interior of a medium
- Using measurements usually acquired at the boundary
Electroencephalography (EEG)

- Non-invasive imaging technique
- Measures electric potentials
- Activity of the functioning brain
- Electrodes at the surface of the scalp
EEG measurements are mostly affected by the **skull** tissue

- Low conductivity
- Complicated structure
- Intra- and extra-subject tissue variability
  - Thickness
  - Geometry

⇒ Need of conductivity estimation techniques
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The inverse conductivity estimation problem

We consider the inverse **skull conductivity estimation** problem using **EEG data**, in the preliminary case of an **homogeneous** skull conductivity.
Simplified model

- Three layer **spherical head** model
- Three concentric **nested spheres** modelling: the scalp $\Omega_2$, skull $\Omega_1$ and brain $\Omega_0$ tissues
- The head is assumed to be **piecewise homogeneous**: each of the three layers is supposed to have a **constant** conductivity

$$\sigma \bigg|_{\Omega_i} = \sigma_i, \quad 0 < \sigma_1 < \min(\sigma_0, \sigma_2)$$
Model for the electric potential $u$

We consider the conductivity Poisson equation:

$$\nabla \cdot (\sigma \nabla u) = \nabla \cdot J^P \text{ in } \mathbb{R}^3,$$

Modeling the primary current $J^P$ as the result of the superposition of $Q$ pointwise dipolar sources:

$$J^P = \sum_{q=1}^{Q} p_q \delta_{C_q},$$

where $\delta_{C_q}$ is the Dirac distribution supported at $C_q \in \Omega_0$. 

Model for the electric potential with $Q$ dipolar sources

$$\nabla \cdot (\sigma \nabla u) = \sum_{q=1}^{Q} p_q \cdot \nabla \delta_{C_q} \text{ in } \mathbb{R}^3.$$
We solve the conductivity estimation problem from the available \textit{partial} boundary EEG data on $S_2$:

\[
\begin{align*}
    u_2 &= g, \text{ \textit{pointwise} EEG values} \\
    \partial_n u_2 &= 0, \text{ no current flux outside the head}
\end{align*}
\]

Assuming that the source term has been already \textit{estimated}
Expansions and boundary conditions

The source activity \( u_0 \) and the EEG data \( g \) are expanded on spherical harmonics basis:

\[
  u_0(r) = \sum_{k,m} [\alpha_{0km} r^k + \beta_{0km} r^{-(k+1)}] Y_{km}(\theta, \phi), \quad r \in \Omega_0 \setminus \{C_q\}
\]

where \( r = (r, \theta, \phi) \), \( k \in \mathbb{N}^* \), \( m \in \mathbb{Z} \), and \(- k \leq m \leq k\).

\[
g(\theta, \phi) = \sum_{k,m} g_{km} Y_{km}(\theta, \phi) \text{ on } S_2
\]

The EEG data are transmitted over the spheres \( S_1, S_0 \) with the boundary conditions:

\[
\begin{cases}
  u_{i-1} = u_i & \text{on } S_i \\
  \sigma_{i-1} \partial_n u_{i-1} = \sigma_i \partial_n u_i & \text{on } S_i
\end{cases}
\]
Matrix representation of the boundary conditions

A solution $u(r)$ in a domain $\Omega_i$ is expanded on spherical harmonics as:

$$u(r) = \sum_{k,m} [\alpha_{ikm} r^k + \beta_{ikm} r^{-(k+1)}] Y_{km}(\theta, \phi), \quad r \in \Omega_i$$

whereas it's outwards normal derivative as:

$$\partial_n u(x) = \partial_r u(r) = \sum_{k,m} [\alpha_{ikm} k r^{k-1} - \beta_{ikm} (k + 1) r^{-(k+2)}] Y_{km}(\theta, \phi)$$

The transmission from $[u_i \quad \partial_n u_i]$ on $S_i$ to $[u_{i-1} \quad \partial_n u_{i-1}]$ on $S_{i-1}$, $i = 1, 2$, can be simplified using:

$$\begin{bmatrix} r^k & r^{-(k+1)} \\ \sigma k r^{k-1} & -\sigma (k + 1) r^{-(k+2)} \end{bmatrix} = T_k(r, \sigma)$$
Computing the **data transmission** over the spherical interfaces, the spherical harmonics coefficients of:

- the EEG measurements: $g_{km}$
- the source term: $\beta_{0km}$

can be linked as:

$$\beta_{0km} = \begin{bmatrix} 0 & 1 \end{bmatrix} T_k^{-1}(r_0, \sigma_0) T_k(r_0, \sigma_1) T_k^{-1}(r_1, \sigma_1) T_k(r_1, \sigma_2) T_k^{-1}(r_2, \sigma_2) \begin{bmatrix} g_{km} \\ 0 \end{bmatrix}$$
Data transmission over the spherical interfaces

Computing the **data transmission** over the spherical interfaces the spherical harmonics coefficients of:

- the EEG measurements: $g_{km}$
- the source term: $\beta_{0km}$

can be linked as:

$$
\beta_{0km} = [0 \ 1] \ T_{k}^{-1}(r_0, \sigma_0) \ T_{k}(r_0, \sigma_1) \ T_{k}^{-1}(r_1, \sigma_1) \ T_{k}(r_1, \sigma_2) \ T_{k}^{-1}(r_2, \sigma_2) \ [g_{km} \ 0]
$$

Solving this equation in terms of $\sigma_1$, leads to a **polynomial equation** $P(\sigma_1) = 0$ of $\text{deg } P = 2$ in $\sigma_1$, depending on:

$P = P_k, r_0, r_1, r_2, \sigma_0, \sigma_2$. 
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The polynomial is of the form:

\[ P(\sigma_1) = B_1(k) \sigma_1 \beta_{0km} - (A_2(k) \sigma_1^2 + A_1(k) \sigma_1 + A_0(k)) \ g_{km} \]

Let \( \sigma_{1,k} \) be the one of the two roots of the polynomial \( P \) for the \( k^{th} \) spherical harmonic element.

The unique admissible solution \( \sigma_{1,k} \), satisfies:

\[ 0 < \sigma_{1,k} < \min(\sigma_0, \sigma_2) \]

and make \( |P| \) achieving its minimal value (\( |P| = 0 \)).
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Reconstruction algorithm

Consider the error function:

\[ \varepsilon_k(\sigma_1, \beta_{0km}, g_{km}) = B_1(k) \sigma_1 \beta_{0km} - (A_2(k) \sigma_1^2 + A_1(k) \sigma_1 + A_0(k)) \ g_{km} \]

As the reconstruction of the conductivity \( \sigma_1 \) does not depend on the spherical harmonics index \( m \), the following normalization is applied over the different spherical harmonics index \( k \):

\[
\begin{align*}
    g_k &= \sum_m g_{km} \bar{\beta}_{km} \\
    \beta_{0k} &= \sum_m \beta_{km} \bar{\beta}_{km} = \sum_m |\beta_{km}|^2
\end{align*}
\]

The estimated conductivity values is computed solving a least square minimization of the error equation for \( K > 0 \):

\[
\sigma_{1}^{est} = \arg \min_s \sum_{k=0}^{K} |\varepsilon_k(s, \beta_{0k}, g_k)|^2
\]
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Numerical analysis

Numerical analysis of the inverse conductivity estimation problem from:

- **EEG measurements** on the scalp $S_2$ and **sources activity**
- Both expanded on the spherical harmonics basis
  - Simulated by the FindSources3D software\(^1\) (FS3D)

Using **known** source activity

✓ Perfect conductivity reconstruction

Using **estimated** sources activity

✓ Good results, depended on the quality of the source estimation

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\(^1\)See [http://www-sop.inria.fr/apics/FindSources3D/](http://www-sop.inria.fr/apics/FindSources3D/).
Stability with respect to the source term

Simulated EEG data by FS3D
- Single dipole $\mathbf{C}_1$ at frontal lobe
  - with moment $\mathbf{p}_1$
  - spherical harmonics coefficients $g_k$ and $\beta_{0k}$
- 20 inexact locations $\mathbf{C}_1^n$ located at a constant distance from $\mathbf{C}_1$
  - same moment $\mathbf{p}_1$
  - spherical harmonics coefficients $\beta_{0k}^n$

Conductivity estimation
- from the pairs $g_k, \beta_{0k}^n$
Influence of source mislocation on conductivity estimation

(a) Conductivity estimation results for various mislocations \( C_1^n \) of the actual dipole \( C_1 \), computed as a percent of the brain radius \( r_0 \)

(b) Relative error for the mean values \( \tilde{\sigma}_1^{est} \) of the estimated conductivities among the 20 mislocations of the original dipole

Conductivities: \( \sigma_0 = \sigma_2 = 0.33 \text{ S/m} \) while \( \sigma_1 = 0.0042 \text{ S/m} \) in simulated EEG data
Radii: \( r_0 = 0.87, r_1 = 0.92 \) and \( r_2 = 1 \). Number of EEG channels: 81
Summary

- Uniqueness of the inverse conductivity problem
- Reconstruction algorithm
  - robustness with respect to the source term

Future plans:
- Extend numerical analysis to various source configurations:
  - locations, moments and orientations
- Restrict the necessary knowledge of the source term
  - measurements
  - other modalities
- Comparison of results with more realistic head models and skull layer: joint work in progress
Thank you for your attention!

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